

SMALL FRACTIONAL PARTS OF POLYNOMIALS

ROGER BAKER

ABSTRACT. Let $k \geq 6$. Using the recent result of Bourgain, Demeter, and Guth [5] on the Vinogradov mean value, we obtain new bounds for small fractional parts of polynomials $\alpha_k n^k + \cdots + \alpha_1 n$ and additive forms $\beta_1 n_1^k + \cdots + \beta_s n_s^k$. Our results improve earlier theorems of Danicic (1957), Cook (1972), Baker (1982, 2000), Vaughan and Wooley (2000), and Wooley (2013).

1. INTRODUCTION

Let $J_{s,k}(N)$ be the Vinogradov mean value,

$$J_{s,k}(N) := \int_{[0,1]^k} \left| \sum_{n=1}^N e(x_k n^k + \cdots + x_1 n) \right|^{2s} dx_1 \dots dx_k.$$

Here s and k are natural numbers. Recently Wooley [12] (for $k = 3$) and Bourgain, Demeter, and Guth [5] (for $k \geq 4$) have established the main conjecture for $J_{s,k}(N)$, namely

$$(1.1) \quad J_{s,k}(N) \ll_{k,\varepsilon} N^{s+\varepsilon} + N^{2s-k(k+1)/2+\varepsilon}.$$

Here ε is an arbitrary positive number. In the present note we combine (1.1) with techniques from two earlier publications [3, 4] to obtain new bounds of the form

$$(i) \quad \min_{1 \leq n \leq N} \|\alpha_k n^k + \cdots + \alpha_1 n\| \ll_{k,\varepsilon} N^{-\mu_k+\varepsilon} \quad (k = 8, 9, \dots)$$

(with arbitrary real numbers $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_s$ here and below);

$$(ii) \quad \min_{1 \leq n \leq N} \|\alpha_k n^k + \alpha_1 n\| \ll_{k,\varepsilon} N^{-\rho_k+\varepsilon} \quad (k = 6, 7, \dots)$$

$$(iii) \quad \min_{\substack{0 \leq n_1, \dots, n_s \leq N \\ (n_1, \dots, n_s) \neq \mathbf{0}}} \|\beta_1 n_1^k + \cdots + \beta_s n_s^k\| \ll N^{-\sigma_{s,k}+\varepsilon} \quad (k = 6, 7, \dots, s \geq 1).$$

Theorem 1. *Let $k \geq 8$. Then (i) holds with $\mu_k = 1/2k(k-1)$.*

2010 *Mathematics Subject Classification.* Primary 11J54.

Theorem 2. (a) Let $k \geq 6$. Then (ii) holds with $\rho_k = 1/k(k-1)$.
 (b) Let $k \geq 6$. For a certain positive absolute constant B , (ii) holds with $\rho_k = 1/k(2 \log k + B \log \log k)$.

Theorem 3. (a) Let $k \geq 6$, $1 \leq s \leq k(k-1)$. Then (iii) holds with $\sigma_{s,k} = s/k(k-1)$.

(b) Let

$$F(J, s, k) = \min \left(\frac{s}{J}, \max_{J+1 \leq h \leq s} \min \left(\frac{(2h-2)(s-k) + 4k-4}{h(s-k) + 4h-4}, \frac{s-h+J+1}{J} \right) \right)$$

Then (iii) holds for $k \geq 6$, $s > k(k-1)$ with

$$\sigma_{s,k} = F(k(k-1), s, k).$$

In particular,

$$\min_{\substack{0 \leq n_1, \dots, n_s \leq N \\ (n_1, \dots, n_s) \neq \mathbf{0}}} \|\beta_1 n_1^6 + \dots + \beta_s n_s^6\| \ll N^{-s/30+\varepsilon} (1 \leq s \leq 56).$$

We note here the existing results in each case. Let $K = 2^{k-1}$.

(i) This is known with $\mu_k = 1/K$ ($2 \leq k \leq 8$) (Baker [1]) and $\mu_k = 1/4k(k-2)$ for $k \geq 9$ (Wooley [11]).

(ii) Only the special case $\alpha_1 = 0$ has been considered separately from (i). Here the result is known with $\rho_2 = 4/7$ (Zaharescu [14]); $\rho_k = 1/K$ ($3 \leq k \leq 6$) (Danicic [7]), while there are the values $\rho_7 = 1/57.23$, $\rho_8 = 1/69.66$, $\rho_9 = 1/82.08$, $\rho_{10} = 1/94.62$, $\rho_{11} = 1/107.27, \dots, \rho_{20} = 1/222.16$, given by Vaughan and Wooley [9], which are better than the present method gives (in the monomial case) for $k \geq 11$. There is an absolute positive constant C such that, for $k \geq 6$,

$$(1.2) \quad \min_{1 \leq n \leq N} \|\alpha n^k\| \ll_{k,\varepsilon} N^{-1/k(\log k + C \log \log k)}$$

(Wooley [10]).

(iii) This is known with $\sigma_{s,k} = s/K$ for $k \geq 2$, $1 \leq s \leq K$ (Cook [6]), and

$$\sigma_{s,k} = F(K, s, k) \quad (k \geq 4, s > K)$$

(Baker [4]). For $k = 2, 3$ and $s > K$, see Baker [1, 4]; for example, $\sigma_{3,2} = 9/8$ and $\sigma_{5,3} = 5/4$.

We refer the reader to Heath-Brown [8], Wooley [10], and Vaughan and Wooley [9] for results of the kind: for irrational α , we have

$$\|\alpha n^k\| < n^{-\tau_k}$$

for infinitely many k . For example, one may take $\tau_k = 1/9.028k$ for every k [10].

2. BOUNDS FOR WEYL SUMS

We suppose throughout (as we may) that ε is sufficiently small and N is sufficiently large in terms of k, ε ; we write $\eta = \varepsilon^2$.

Theorem 4. *Let $k \geq 3$ and $\varepsilon > 0$. Suppose that the Weyl sum*

$$g_k(\boldsymbol{\alpha}; N) := \sum_{n=1}^N e(\alpha_k n^k + \cdots + \alpha_1 n)$$

satisfies

$$(2.1) \quad |g_k(\boldsymbol{\alpha}; N)| \geq A > N^{1-1/2k(k-1)+\varepsilon}.$$

Then there exist integers q, a_1, \dots, a_k such that

$$(2.2) \quad 1 \leq q \leq N^\varepsilon (NA^{-1})^k$$

and

$$(2.3) \quad |q\alpha_j - a_j| \leq N^{-j+\varepsilon} (NA^{-1})^k \quad (1 \leq j \leq k).$$

If $\alpha_{k-1} = \cdots = \alpha_2 = 0$, then the same conclusion holds with the weaker lower bound.

$$(2.4) \quad |g_k(\boldsymbol{\alpha}; N)| \geq A > N^{1-1/k(k-1)+\varepsilon}$$

in place of (2.1).

Proof. We initially proceed exactly as in the proof of [3, Theorem 4.3] with θ replaced by 0 and ℓ replaced by $(k-1)/2$. This is permissible since we have

$$J_{s,k-1}(N) \ll N^{s+\varepsilon}$$

with $s = k(k-1)/2$, in place of the bound for $J_{s,k-1}(N)$ used in [3]. We find that for $j = 2, \dots, k$ there are coprime pairs of integers q_j, b_j with

$$1 \leq q_j \ll (NA^{-1})^{k(k-1)} (\log N)^C$$

$$|q\alpha_j - b_j| \leq N^{-j+\varepsilon} (NA^{-1})^{k(k-1)}$$

where we shall use C for an unspecified positive constant depending on k . Let q_0 be the l.c.m of q_2, \dots, q_k . We now follow the argument of [3, pp. 41–42] to obtain

$$(2.5) \quad q_0 \ll (\log N)^C (NA^{-1})^{k(k-1)}.$$

It follows that, with $a_j = q_0 b_j / q_j$, we have

$$(2.6) \quad |q_0 \alpha_j - a_j| \leq N^{-j+2\varepsilon} (NA^{-1})^{2k(k-1)} \quad (j = 2, \dots, k).$$

We now appeal to Lemma 4.6 of [3], which we restate here for clarity as Lemma 1.

Lemma 1. *Suppose that there are integers r, v_2, \dots, v_k such that $\gcd(r, v_2, \dots, v_k) = 1$,*

$$(2.7) \quad |q_j r - v_j| \leq N^{1-j}/4k^4 \quad (j = 2, \dots, k),$$

and that

$$(2.8) \quad |g_k(\alpha; N)| \geq H > r^{1-1/k} N^\varepsilon.$$

There is a natural number $t \leq 2k^2$ such that

$$(2.9) \quad tr \leq (NH^{-1})^k N^\varepsilon,$$

$$(2.10) \quad t|\alpha_j r - v_j| \leq (NH^{-1})^k N^{-j+\varepsilon} \quad (j = 2, \dots, k)$$

$$(2.11) \quad \|tr \alpha_1\| \leq (NH^{-1}) N^{-1+\varepsilon}.$$

We now apply the lemma with $A = H$, $r = q_0 d^{-1}$, $v_j = a_j d^{-1}$ where $d = \gcd(q_0, a_2, \dots, a_k)$. From (2.5) and (2.6),

$$|\alpha_j r - v_j| \leq N^{-j+2\varepsilon} (NA^{-1})^{2k(k-1)} \leq N^{-j+1} (4k^4)^{-1}$$

since

$$(NA^{-1})^{2k(k-1)} \leq N^{1-12\varepsilon}$$

and $r \leq N^{1-5\varepsilon}$,

$$Ar^{-1+1/k} N^{-2\varepsilon} \geq N^{1-1/k(k-1)-1+1/k-C\varepsilon} \gg 1.$$

The inequalities (2.9)–(2.11) now yield the first assertion of the theorem with $q = tr$. For the second assertion, since $\alpha_2, \dots, \alpha_{k-1}$ are 0, we may take $r = q_k$, $v_k = b_k$, $v_2 = \dots = v_{k-1} = 0$, $H = A$ in the application of Lemma 1. (The inequality (2.4) suffices in the earlier part of the argument.) We know that

$$|r\alpha_k - a_k| \leq N^{-k+\varepsilon} (NA^{-1})^{k(k-1)}$$

rather than the weaker bound (2.6). We may now complete the proof in the same way as before. \square

3. PROOF OF THEOREMS 1, 2, AND 3

Proof of Theorem 1. Suppose there is no solution of

$$(3.1) \quad 1 \leq n \leq N, \quad \|\alpha_k n^k + \dots + \alpha_1 n\| \leq N^{-1/J+\varepsilon}$$

where J denotes $2k(k-1)$. By [3, Theorem 2.2] we have

$$\sum_{m=1}^M |g_k(m\alpha; N)| > N/6,$$

where $M = \lceil N^{1/J-\varepsilon} \rceil$. There is an integer m , $1 \leq m \leq M$ such that

$$|g_k(m\alpha; N)| > A = N/6M.$$

We have

$$(NA^{-1})^{2k(k-1)} \ll M^{2k(k-1)} \ll N^{1-2k(k-1)\varepsilon}.$$

By Theorem 4 there is a natural number $q = tr$ such that

$$(3.2) \quad q \ll N^\varepsilon (NA^{-1})^k \ll M^k,$$

$$(3.3) \quad \begin{aligned} \|qm\alpha_j\| &\ll (NA^{-1})^k N^{-j+\varepsilon} \\ &\ll M^k N^{-j+\varepsilon} \quad (j = 1, \dots, k). \end{aligned}$$

Now let $n = qm$. Then

$$\begin{aligned} n &\ll M^{k+1} \ll N^{(k+1)/J} \ll N^{1-\varepsilon}, \\ \|n^j \alpha_j\| &\leq n^{j-1} \|n \alpha_j\| \\ &\ll M^{(k+1)(j-1)+k} N^{-j+\varepsilon} \ll M^{-1} N^{-\varepsilon} \end{aligned}$$

since $M^{(k+1)j} \ll N^{(k+1)j/J-(k+1)\varepsilon} \ll N^{j-2\varepsilon}$. It follows that n satisfies (3.1), which is a contradiction. This completes the proof of Theorem 1. \square

Proof of Theorem 2(a). We follow the above proof; this time, J denotes $k(k-1)$. The second assertion of Theorem 4 provides an integer $q = tr$ satisfying (3.2), and (3.3) for the relevant values $j = 1, k$. Now we complete the proof as before.

Proof of Theorem 2(b). This is a simple consequence of Wooley's bound (1.2). Let $\nu = \nu(k)$ have the property that

$$\min_{1 \leq n \leq N} \|\alpha n^k\| \ll_k N^{-\nu}$$

for $N \geq 1$ and real α . Let $a = \frac{1}{2+\nu}$, $b = 1 - a$. By Dirichlet's theorem there is a natural number $\ell \leq N^b$ with

$$\|\alpha_1 \ell\| \leq N^{-b}.$$

We now choose another natural number $m \leq N^a$ with

$$\|\alpha_k \ell^k m^k\| \ll N^{-a\nu} = N^{-\nu/(2+\nu)}.$$

Note that

$$\|\alpha_1 \ell m\| \leq N^{a-b} = N^{2a-1}.$$

Since $2a - 1 = -\frac{\nu}{2+\nu}$, we have, with $n = \ell m$,

$$1 \leq n \leq N, \quad \|\alpha_k n^k + \alpha_1 n\| \ll N^{-\nu/(2+\nu)}.$$

Taking $\nu = 1/k(\log k + C \log \log k)$, we obtain

$$\frac{\nu}{2 + \nu} = \frac{1}{2k \log k + 2C \log \log k + 1},$$

so that Theorem 2(b) holds with a suitable choice of B . \square

Example. If we take $k = 20$, $\nu = 1/222.16$ from [9], we obtain the value $1/445.32$ for ρ_{20} , which is not as good as Theorem 2(a). The proof of Theorem 2(b) is relatively crude, so it may be possible to do better using ideas from [9], [10].

Proof of Theorem 3(b). We can follow the proof of Theorem 1.8 of [4] (in the case $k \geq 4$) verbatim, replacing K by $J := k(k-1)$. The role of Lemma 5.2 of [4] is played by Theorem 4 in conjunction with [3, Lemma 8.6].

Proof of Theorem 3(a). Write $J = k(k-1)$ again. We assume that there is no solution of

$$(3.4) \quad \|\beta_1 n_1^k + \cdots + \beta_s n_s^k\| \leq N^{-s/J+\varepsilon}$$

with $0 \leq n_1, \dots, n_s \leq N$, $(n_1, \dots, n_s) \neq \mathbf{0}$. Let

$$S_i(m) = \sum_{n=1}^N e(m\beta_i n^k), \quad L = \lfloor N^{1/J-\varepsilon} \rfloor.$$

Following [4], Lemma 5.1, we find that there is a set \mathcal{B} of natural numbers, $\mathcal{B} \subset [1, L]$, and there are positive numbers $B_1 \geq \cdots \geq B_s$ such that

$$B_i < |S_i(m)| \leq 2B_i \quad (i = 1, \dots, s)$$

and

$$B_1 \dots B_s |\mathcal{B}| \gg N^{s-\eta}.$$

(This may require a reordering of β_1, \dots, β_s .) We can now follow the proof of Lemma 5.4 on [4], with K replaced by J , to obtain the inequality

$$|\mathcal{B}| \ll LN^{-1+2k\eta} |\mathcal{B}|^{k/s}.$$

Suppose first that $s > k$. Then

$$LN^{-1+2k\eta} \gg |\mathcal{B}|^{1-k/s} \gg 1,$$

contrary to the definition of L .

Suppose now that $s \leq k$. Then

$$L^{\frac{k}{s}-1} \geq |\mathcal{B}|^{\frac{k}{s}-1} \gg L^{-1} N^{1-2k\eta},$$

$$L \gg N^{\frac{s}{k}-2s\eta}.$$

This is again contrary to the definition of L , and we conclude that there is a solution of (3.4). \square

REFERENCES

- [1] R. C. Baker, Weyl sums and Diophantine approximation, *J. London Math. Soc.* (2) **25** (1982), 25–34. Correction, *ibid.* **46** (1992), 202–204.
- [2] R. C. Baker, Small solutions of congruences, *Mathematika* **30** (1983), 164–188.
- [3] R. C. Baker, *Diophantine Inequalities*, London Mathematical Society Monographs, New Series, vol. 1, Oxford University Press, Oxford, 1986.
- [4] R. C. Baker, Small solutions of congruences, II, *Funct. et Approx. Comment. Math.* **28** (2001), 19–34.
- [5] J. Bourgain, C. Demeter, and L. Guth, Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three, arXiv:1512.01565.
- [6] R. J. Cook, The fractional parts of an additive form, *Proc. Camb. Phil. Soc.* **72** (1972), 209–212.
- [7] I. Danicic, *Contributions to number theory*, Ph.D. thesis, University of London, 1957.
- [8] D. R. Heath-Brown, The fractional part of αn^k , *Mathematika* **35** (1988), 28–37.
- [9] R. C. Vaughan and T. D. Wooley, Further improvements in Waring’s problem, IV: Higher powers, *Acta Arith.* **94** (2000), 203–285.
- [10] T. D. Wooley, The application of a new mean value theorem to fractional parts of polynomials, *Acta Arith.* **65** (1993), 163–179.
- [11] T. D. Wooley, New estimates for smooth Weyl sums, *J. London Math. Soc.* (2) **51** (1995), 1–13.
- [12] T. D. Wooley, Vinogradov’s mean value theorem via efficient congruencing, II, *Duke Math. J.* **162** (2013), 673–730.
- [13] T. D. Wooley, The cubic case of the main conjecture in Vinogradov’s mean value theorem, arXiv:1401.3150.
- [14] A. Zaharescu, Small values of $n^2\alpha \pmod{1}$, *Invent. Math.* **121** (1995), 379–388.